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Classical and Quantum Aspects of Bianchi Type IX Cosmological Model

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A detailed study of the Bianchi type IX cosmological model is done. Classical field equations are discussed with a massive scalar field. The Wheeler–DeWitt quantum equation is formed and is solved using a Born–Oppenheimer type of approximation.

1. INTRODUCTION

So far, isotropic cosmological models have been studied extensively from both classical and quantum standpoints. But is little work on anisotropic models, especially on the Bianchi-IX closed cosmological model, due to the complicated nature of the field equations. Actually, the study of anisotropic models was started after the discovery of the microwave background radiation in 1965. It was found that the radiation was isotropic to one part in 10^4 apart from a dipole anisotropy which was attributed to the peculiar motion of our galaxy (Hawking and Luttrell, 1984).

In this paper the Bianchi-IX cosmological model is studied in detail. The classical field equations are studied for different choices of the variables and the metric on superspace is interpreted physically and also the geodesics on superspace are calculated in Section 2. In Section 3, the Wheeler–DeWitt equation for Einstein–Hilbert action is simplified to a diagonal form by a suitable transformation of the minisuperspace variables and is solved with a Born–Oppenheimer type of approximation. A brief conclusion and future prospects are given in Section 4.

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2. BIANCHI-IX COSMOLOGICAL MODEL WITH A MASSIVE SCALAR FIELD

A Bianchi-IX model has a homogeneous anisotropic closed metric (Hawking and Luttrell, 1984)

$$dS^2 = -N^2(t)\, dt^2 + e^{2\alpha(t)}[e^{2\beta}]_{ij}\omega^i \cdot \omega^j \tag{2.1}$$

where $\beta_{ij}^{(t)}$ is a 3×3 symmetric trace-free matrix and ω^i are three one-forms on the three-sphere which obey

$$d\omega^i = \varepsilon_{ijk}\omega^j \wedge \omega^k$$

A typical explicit form of the metric is (Martens and Nel, 1978; Banerjee and Santos, 1984)

$$dS^2 = -N^2(t)\, dt^2 + a^2(t)\, dx^2 + b^2(t)\, dy^2 + (b^2 \sin^2 y + a^2 \cos^2 y)\, dz^2 - 2a^2 \cos y\, dx\, dz \tag{2.2}$$

Here an overall prefactor is neglected and a constant factor equal to the three-volume is omitted, as they are irrelevant in classical analysis. We have taken a minimally coupled free homogeneous scalar field $\phi(t)$ of mass m. The action of the system is

$$S = \int L \cdot dt$$

where

$$L = \frac{N}{2}\left(a - \frac{1}{4}\frac{a^3}{b^2} - \frac{a\dot{b}^2}{N^2} - \frac{2\dot{a}\cdot\dot{b}\cdot b}{N^2} - ab^2m^2\phi^2 + ab^2\frac{\dot{\phi}^2}{N^2}\right) \tag{2.3}$$

Now variation of S with respect to the lapse function gives the constraint equation

$$\frac{2}{N^2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{1}{N^2}\frac{\dot{b}^2}{b^2} + \frac{1}{b^2} - \frac{1}{4}\frac{a^2}{b^4} - \frac{\dot{\phi}^2}{N^2} - m^2\phi^2 = 0 \tag{2.4}$$

We now parametrize the time scale so that $N(t) = 1$, which will be taken henceforth. The field equations are (Banerjee *et al.*, 1990) (obtained by variation of a, b, and ϕ)

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{1}{4}\frac{a^2}{b^4} + \dot{\phi}^2 - m^2\phi^2 = 0 \tag{2.5}$$

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} - \frac{3}{4}\frac{a^2}{b^4} + \dot{\phi}^2 - m^2\phi^2 = 0 \tag{2.6}$$

$$\ddot{\phi} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\dot{\phi} + m^2\phi = 0 \tag{2.7}$$

If we introduce

$$\alpha = \ln a, \qquad \beta = \ln b$$

then the field equations (2.4)-(2.7) are simplified to

$$2\dot{\alpha}\dot{\beta} + \dot{\beta}^2 + e^{-2\beta} - \tfrac{1}{4} e^{2\alpha - 4\beta} - \dot{\phi}^2 - m^2\phi^2 = 0 \tag{2.8}$$

$$\ddot{\alpha} + \ddot{\beta} + \dot{\alpha}^2 + \dot{\beta}^2 + \dot{\alpha}\dot{\beta} + \tfrac{1}{4} e^{2\alpha - 4\beta} + \dot{\phi}^2 - m^2\phi^2 = 0 \tag{2.9}$$

$$2\ddot{\beta} + 3\dot{\beta}^2 + e^{-2\beta} - \tfrac{3}{4} e^{2\alpha - 4\beta} + \dot{\phi}^2 - m^2\phi^2 = 0 \tag{2.10}$$

$$\ddot{\phi} + (\dot{\alpha} + 2\dot{\beta})\dot{\phi} + m^2\phi = 0 \tag{2.11}$$

The dimensionless variables (Page, 1987; Hawking and Page, 1988)

$$\begin{gathered} x \equiv \phi, \qquad y = m^{-1}\dot{\phi}, \qquad z = m^{-1}\dot{\alpha}, \qquad u = m^{-1}\dot{\beta}, \\ v = \alpha - \beta, \qquad \eta = mt \end{gathered} \tag{2.12}$$

change the above second-order equations to a set of coupled first-order equations:

$$\frac{dx}{d\eta} = y \tag{2.13}$$

$$\frac{dy}{d\eta} = -x - y(z + 2u) \tag{2.14}$$

$$\frac{dz}{d\eta} = \frac{1 + k/4}{l} x^2 - \frac{1}{l} y^2 + \frac{k/2 - 1}{l} u^2 - z^2 - \frac{1 - 3/4k}{l} zu \tag{2.15}$$

$$\frac{du}{d\eta} = \frac{k}{4l} x^2 - \frac{1 - k/2}{l} y^2 - \frac{1}{l} u^2 + \frac{1 - 3/4k}{l} zu \tag{2.16}$$

$$\frac{dv}{d\eta} = z - u \tag{2.17}$$

with $k = e^{2v}$, $l = 1 - k/4$. The first integral of the above set of first-order equations, i.e., the constraint equation, takes the form

$$\frac{l}{m^2 b^2} = x^2 + y^2 - 2zu - u^2 \tag{2.18}$$

Thus, the solutions represent four-parameter congruence of trajectories in the five-dimensional (x, y, z, u, v) space.

We may further study the field equations in the three-dimensional minisuperspace parametrized by the coordinates (a, b, ϕ). They are nothing but the timelike geodesics in the three-dimensional auxiliary metric (Fang and Ruffini, 1987)

$$ds^2 = M^2(dr^2 + d\phi^2 - d\delta^2) \tag{2.19}$$

where

$$M^2 = m^2\phi^2\, e^{2(\gamma+9\delta)} + \tfrac{1}{4}\, e^{4(\gamma+\delta)} - e^{2(\gamma+5\delta)} \tag{2.20}$$

with

$$\gamma = \alpha - \beta/4, \qquad \delta = \beta/4$$

Because the auxiliary metric (2.19) is conformally flat, the trajectories may also be interpreted as those of a particle (Page, 1984) of variable mass squared M^2 moving in the flat three-dimensional Minkowski metric $-d\delta^2 + d\gamma^2 + d\phi^2$.

The geodesic equations for the auxiliary metric (2.19) may be written as a set of two second-order equations (Page, 1984) (eliminating the affine parameter):

$$\frac{d^2\gamma}{d\delta^2} + \frac{1}{M^2}\left[1 - \left(\frac{d\gamma}{d\delta}\right)^2 - \left(\frac{d\phi}{d\delta}\right)^2\right]\left(e + f\frac{d\gamma}{d\delta}\right) = 0 \tag{2.21a}$$

$$\frac{d^2\phi}{d\delta^2} + \frac{1}{M^2}\left[1 - \left(\frac{d\gamma}{d\delta}\right)^2 - \left(\frac{d\phi}{d\delta}\right)^2\right]\left(g + f\frac{d\phi}{d\delta}\right) = 0 \tag{2.21b}$$

where

$$e = m^2\phi^2\, e^{2(\gamma+9\delta)} + \tfrac{1}{2}\, e^{4(\gamma+\delta)} - e^{2(\gamma+5\delta)}$$

$$f = 9m^2\phi^2\, e^{2(\gamma+9\delta)} + e^{4(\gamma+\delta)} - 5\, e^{2(\gamma+5\delta)}$$

$$g = m^2\phi\, e^{2(\gamma+9\delta)}$$

For a massless scalar field, combining (2.21a) and (2.21b), we have that the equation for geodesics reduces to

$$\left(\mu + \frac{d\gamma}{d\delta}\right)\frac{d^2\phi}{d\delta^2} = \frac{d\phi}{d\delta}\frac{d^2\gamma}{d\delta^2} \tag{2.22}$$

with

$$\mu = \left.\frac{e}{f}\right|_{m=0}$$

Thus, the auxiliary metric (2.18) and the geodesics (2.21a)-(2.21b) are singular corresponding to the points $M^2 = 0$, but the trajectories simply pass through these curves in the configuration space without any singularity in the physical metric (2.2) (Page, 1984).

3. WHEELER–DEWITT EQUATION AND ITS SOLUTION

Let us now return to the Lagrangian (2.3); the corresponding canonical momenta are

$$p_a = -b\dot{b}, \qquad p_b = -(a\dot{b} + b\dot{a}), \qquad p_\phi = ab^2\phi$$

So the Hamiltonian in terms of canonically conjugate variables (a, p_a), (b, p_b), and (ϕ, p_ϕ) is

$$H = \frac{1}{2}\left(\frac{a}{b^2}p_a^2 - \frac{1}{b}p_a \cdot p_b + \frac{1}{ab^2}\dot{p}_\phi^2 - a + \frac{1}{4}\frac{a^3}{b^2} + m^2\phi^2 ab^2\right) \tag{3.1}$$

Hence, replacing the momenta by the corresponding operators the Wheeler-DeWitt equation is (Fang and Ruffini, 1987)

$$H \cdot \psi = 0$$

So with a particular choice of operator ordering, the explicit form will be

$$2ab\frac{\partial^2\psi}{\partial a\,\partial b} - a^2\frac{\partial^2\psi}{\partial a^2} - \frac{\partial^2\psi}{\partial\phi^2} - a^2b^2\psi + \frac{a^4 \cdot \psi}{4} + m^2\phi^2a^2b^4\psi = 0 \tag{3.2}$$

By the change of variables

$$\alpha = \ln a, \qquad \beta = \ln(a \cdot b) \tag{3.3}$$

we can diagonalize the D'Alembertian and the resulting equation is

$$\frac{\partial^2\psi}{\partial\beta^2} - \frac{\partial^2\psi}{\partial\alpha^2} - \frac{\partial^2\psi}{\partial\phi^2} - e^{2\beta} \cdot \psi + \frac{e^{4\alpha}}{4} \cdot \psi + m^2\phi^2 \cdot e^{2(2\beta-\alpha)}\psi = 0 \tag{3.4}$$

This is a second-order hyperbolic equation in the three-dimensional (α, β, ϕ) space.

We shall now solve this partial differential equation with some approximation. The separation-of-variables method will not be applicable because of the term containing mass of the scalar field. So we use a Born–Oppenheimer (Kiefer, 1987, 1988; Hartle, 1987; Brout *et al.*, 1987) type approximation where adiabatic expansion of the wave function is done. Accordingly, we use the ansatz (Kiefer, 1987, 1988)

$$\psi(\alpha, \beta, \phi) = \sum_n C_n(\alpha, \beta)\Phi_n(\alpha, \beta, \phi) \tag{3.5}$$

$\Phi_n(\alpha, \beta, \phi)$ are the oscillatory eigenfunctions of the eigenvalue equation for the harmonic oscillator in ϕ (assuming Φ_n depends adiabatically on α, β)

$$\left[-\frac{\partial^2}{\partial\phi^2} + \omega^2(\alpha, \beta)\phi^2 - e^{2\beta} + \frac{e^{4\alpha}}{4}\right]\Phi_n(\alpha, \beta, \phi)$$
$$= E_n(\alpha, \beta)\Phi_n(\alpha, \beta, \phi) \tag{3.6}$$

The explicit forms of the frequency, energy eigenvalues, and eigenfunctions are

$$\omega(\alpha, \beta) = m\, e^{2\beta-\alpha}$$

$$E_n(\alpha, \beta) = (2n+1)m\, e^{2\beta-\alpha} - e^{2\beta} + e^{4\alpha}/4$$

$$\Phi_n(\alpha, \beta, \phi) = \left[\frac{\omega(\alpha, \beta)}{\pi}\right]^{1/4} \frac{1}{(2^n \cdot n!)^{1/2}} H_n(\phi[\omega(\alpha, \beta)]^{1/2})\, e^{-[\omega(\alpha,\beta)/2]\phi^2}$$

Substituting (3.5) into (3.4), we obtain

$$\sum_n \Phi_n \left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + E_n(\alpha, \beta)\right) C_n(\alpha, \beta) + 2\sum_n \left(\frac{\partial c_n}{\partial\beta}\frac{\partial\Phi}{\partial\beta} - \frac{\partial c_n}{\partial\alpha}\frac{\partial\Phi}{\partial\alpha}\right)$$

$$+\sum C_n \left(\frac{\partial^2\Phi_n}{\partial\beta^2} - \frac{\partial^2\Phi_n}{\partial\alpha^2}\right) = 0$$

Taking the scalar product with Φ_l and using the orthonormal property of the eigenfunctions $\{\Phi_n\}$ and equation (3.6), one gets

$$\left[\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + E_l(\alpha, \beta)\right] C_l(\alpha, \beta) = 0 \tag{3.7}$$

where the coupled terms are neglected, according to this method. Taking α as the adiabatic parameter, a similar expansion for $C_l(\alpha, \beta)$ is

$$C_l(\alpha, \beta) = \sum_k U_k^{(l)}(\alpha) v_k^{(l)}(\alpha, \beta) \tag{3.8}$$

Here $v_k^{(l)}(\alpha, \beta)$ satisfies the differential equation

$$\frac{\partial^2 v_k}{\partial\beta^2} + e^{2\beta} A(\alpha) v_k = K^2 v_k$$

(the superscript is omitted) with

$$A(\alpha) = (2n+1)m\, e^{-\alpha} - 1$$

Hence, the solution for v_k is

$$v_k(\alpha, \beta) = J_k(e^{\beta}[A(\alpha)]^{1/2}) \quad \text{or} \quad I_k(e^{\beta}|A(\alpha)|^{1/2})$$

according as $A(\alpha) >$ or < 0. Similarly, one has $u_k = K_{k/2}(e^{2\alpha}/4)$, the modified Bessel function. So the general solution is

$$\psi(\alpha, \beta, \phi) = \sum_n \sum_k \left[\frac{\omega(\alpha \cdot \beta)}{\pi}\right]^{1/4} \frac{1}{(2^n \cdot n!)^{1/2}} H_n(\phi[\omega(\alpha, \beta)]^{1/2})$$

$$\times e^{-[\omega(\alpha,\beta)/2]/\phi^2} B_k(e^{\beta}|A(\alpha)|^{1/2}) K_{k/2}(e^{2\alpha}/4) \tag{3.9}$$

where B_k is the Bessel or modified Bessel function of order k and H_n is the Hermite polynomial of degree n.

Moreover, if the scalar field is assumed to be massless, then equation (3.4) simplifies to

$$\frac{\partial^2 \psi}{\partial \beta^2} - \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{\partial^2 \psi}{\partial \phi^2} - e^{2\beta}\psi + \frac{e^{4\alpha}}{4}\psi = 0 \tag{3.10}$$

where a separation ansatz is applicable. So we write

$$\psi(\alpha, \beta, \phi) = p(\alpha) \cdot Q(\beta) \cdot R(\phi) \tag{3.11}$$

The differential equations for $p(\alpha)$, $Q(\beta)$, and $R(\phi)$ are

$$\frac{d^2 p}{d\alpha^2} - \frac{e^{4\alpha} p}{4} - p^2 P = 0$$

$$\frac{d^2 \phi}{d\beta^2} - e^{2\beta} Q - q^2 Q = 0$$

$$\frac{d^2 R}{d\phi^2} - r^2 R = 0$$

(p, q, r are arbitrary constants). Hence, the explicit form of ψ is

$$\psi(\alpha, \beta, \phi) = K_{p/2}(e^{2\alpha}/4) K_q(e^{\beta}) \exp(-r\phi)$$

Hence, $\psi \to 0$ for large volume, i.e., the wave function remains finite even if the universe expands infinitely.

4. CONCLUSION

The correct behavior of the wave function (exponential or oscillatory in form) in the classically forbidden region or classically allowed region is obtained from the solution (4.9) due to the asymptotic behavior of the modified Bessel function or Bessel function. Thus, a massive scalar field model in the Bianchi-IX ansatz may predict correct behavior in both regions. But the solutions for a massless scalar field predict only the classically forbidden region, as the asymptotic behavior is only exponential in form. So the massive scalar field model is physically more interesting than the massless scalar field.

It is interesting to determine the wave function using the proposal of Hartle and Hawking (1983) with the recently developed concept of micro-superspace (Halliwell and Louko, 1988*a*,*b*; Hartle, 1989) and compare it with the above wave function, derived from the Wheeler–DeWitt equation.

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